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BOSTON UNIVERSITY

GRADUATE SCHOOL

THESIS

THE EXPONENTIAL FUNCTION

by

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THE EXPONENTIAL FUNCTION

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The Exponential Function

1. Definition of an exponent.

a^n = a.a.a - - - to n factors where n is a positive integer.

2. The laws of exponents.

Theorem 1. $a^m . a^n = a^{m+n}$

For $a^m . a^n$ = (a.a.a -- to m factors.)(a.a.a --- to n factors) or a.a.a ---- to m+n factors which equals a^{m+n}

Theorem 2. $(a^m)^n = a^{mn}$

For $(a^m)^n = a^m . a^m . a^m$ - - - - to n factors

$= a^{m+m+m}$ ---- n times

$= a^{mn}$

Theorem 3. $a^m \div a^n = a^{m-n}$ for m greater than n

for $a^m = a^{m-n} . a^n$

Then $\frac{a^m}{a^n} = \frac{a^{m-n} . a^n}{a^n} = a^{m-n}$

When n is greater than m

$$\frac{a^m}{a^n} = \frac{a^m}{a^{n-m} . a^m} = \frac{1}{a^{n-m}}$$

Theorem 4. $(ab)^n = a^n b^n$

for $(ab)^n$ = ab . ab. ab. --- to n factors

= (a.a.a ----- to n factors).(b.b.b --- to n factors)

$= a^n b^n$

Reference: College Algebra by H. B. Fine

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3. Definitions of fractional, zero and negative exponents.

The fractional exponent

$a^{p/q} = \sqrt[q]{a^p}$

For $(a^{p/q})^q = a^{p/q} \cdot a^{p/q} \cdot a^{p/q}$ ----- to q factors

$= a^{p/q+p/q+p/q}$ ------ to q terms

$= a^p$ That is, $a^{p/q}$ is the q th root of a^p.

The zero exponent

$a^o = 1$

For $a^o \cdot a^m = a^{o+m} = a^m$

Therefore $a^o = \frac{a^m}{a^m} = 1$

The negative exponent

$a^{-s} = \frac{1}{a^s}$

For $a^{-s} \cdot a^s = a^{-s+s} = a^o = 1$

Therefore $a^{-s} = \frac{1}{a^s}$

Reference: College Algebra by H. B. Fine

4. Proof that the laws of exponents hold for all rational exponents.

Theorem 1. The law $a^m . a^n = a^{m+n}$ holds good for all rational values of m and n.

Let p, q, r, s be any positive integers

Case 1. When $m = p/q$, $n = r/s$

$$a^{p/q} . a^{r/s} = \sqrt[q]{a^p} . \sqrt[s]{a^r} = \sqrt[qs]{a^{ps}} . \sqrt[qs]{a^{rq}} = \sqrt[qs]{a^{ps+rq}} =$$

$$a^{\frac{ps+rq}{qs}} = a^{p/q+r/s}$$

Case 2. When $m = -p/q$ and $n = -r/s$

By Case 1 and the definition of a negative exponent

$$a^{-p/q} . a^{-r/s} = \frac{1}{a^{p/q} . a^{r/s}} = \frac{1}{a^{p/q+r/s}} = a^{-p/q+(-r/s)}$$

In the same way it can be shown that the law holds good when $m = p/q$, $n = -r/s$

Theorem 2. The law $(a^m)^n = a^{mn}$ holds good for all rational values of m and n

Case 1. When $n = p/q$ where p and q are positive integers.

$$(a^m)^{p/q} = \sqrt[q]{(a^m)^p} = \sqrt[q]{a^{mp}} = a^{\frac{mp}{q}} = a^{m.p/q}$$

Case 2. When $n = -s$, where s is any positive rational number.

$$(a^m)^{-s} = \frac{1}{(a^m)^s} = \frac{1}{a^{ms}} = a^{-ms}$$

Reference: College Algebra by H. B. Fine

Theorem 3. The law $(ab)^n = a^n b^n$ holds good for all rational values of n

Case 1. Let $n = p/q$ where p and q are positive integers

Then $(ab)^{p/q} = \sqrt[q]{(ab)^p} = \sqrt[q]{a^p b^p} = \sqrt[q]{a^p}\sqrt[q]{b^p} = a^{p/q} b^{p/q}$

Case 2. Let $n = -s$ where s is any positive rational number.

Then $(ab)^{-s} = \frac{1}{(ab)^s} = \frac{1}{a^s b^s} = a^{-s} b^{-s}$

5. Definition of an irrational exponent.

When b is irrational a^b is defined as the limit which a^x will approach when x is made to approach b through any sequence of irrational values.

Before proving that this limit is a^b we shall first prove two preliminary theorems concerning exponents.

Theorem 1. If b be a rational number and x be a variable which approaches b through rational values, then

$$\lim_{x \to b} a^x = a^b .$$

Case 1. The theorem holds true when b , the limit of x , is zero.

Select a variable n which takes integral values only and such that we shall always have $-1/n < x < 1/n$ and that when $x \to o$ then $n = \infty$.

Reference: College Algebra by H. B. Fine

Then a^x will always lie between $a^{1/n}$ and $a^{-1/n}$ and since $\lim_{n\to\infty} a^{\frac{1}{n}} = \lim_{n\to\infty} a^{-1/n} = 1$, we have

$$\lim_{x\to o} a^x = 1 = a^o$$

Case 2. The theorem holds true when $b \neq o$

For since $a^x = a^b \cdot a^{x-b}$ then

$\lim_{x\to b} a^x = a^b \cdot \lim_{x\to b} a^{x-b} = a^b$ by case 1.

Theorem 2. If b be an irrational number and x a variable which approaches b through rational values, then a^x will approach a limit as $x \to b$, and the value of this limit is independent of the values which x takes in approaching b.

The reasoning is the same whether $a > 1$ or $a < 1$. We shall consider $a > 1$.

There are infinitely many sequences of rational values through which x may run in approaching b as limit. From them select some particular increasing sequence, and represent x by x' when supposed to run through this sequence. Then as $x' \to b$, the variable $a^{x'}$ continually increases but remains finite, less, for instance, than a^c, if c denote any rational greater than b. Hence $a^{x'}$ approaches a limit. Let this limit be L.

Reference: College Algebra by H. B. Fine

It remains to prove that a^x will approach this same limit L if x approaches b through any other sequence of rationals than that through which x' runs.

But $a^x = a^{x'} \cdot a^{x-x'}$ and therefore

$$\lim_{x\to b} a^x = \lim_{x'\to b} a^{x'} \cdot \lim_{x\to b,\ x'\to b} a^{x-x'}$$

Since $\lim_{x\to b\ x'\to b} a^{x-x'} = 1$ by theorem 1

Then $\lim_{x\to b} a^x = \lim_{x'\to b} a^{x'} = L$

To prove that $\lim_{x\to b} a^x = a^b$ when x approaches b through a sequence of irrational values.

Let x', x , x'' denote variables all of which approach b as limit, x' and x'' through sequences of rational values and x through a sequence of irrational values, and such that $x' < x < x''$. Then a^x lies between $a^{x'}$ and $a^{x''}$

By theorems 2 and 1

$$\lim_{x'\to b} a^{x'} = \lim_{x''\to b} a^{x''} = a^b$$

Then $\lim_{x\to b} a^x = a^b$

Reference: College Algebra by H. B. Fine

6. Proof that the laws of exponents are valid for irrational exponents.

Let b and c denote irrational numbers and x and y variables which approach b and c as limits. We suppose x and y to take rational values only.

To show that $a^b . a^c = a^{b+c}$

Since $a^x . a^y = a^{x+y}$, then $\lim a^x a^y = \lim a^{x+y}$

But $\lim a^x a^y = \lim a^x . \lim a^y = a^b a^c$

and $\lim a^{x+y} = a^{\lim(x+y)} = a^{b+c}$

Therefore $a^b . a^c = a^{b+c}$

To show that $(a^b)^c = a^{bc}$

$(a^x)^y = a^{xy}$

* $\lim\limits_{x \to b} (a^x)^y = \lim\limits_{x \to b} a^{xy}$

or $(a^b)^y = a^{by}$

$\lim\limits_{y \to c} (a^b)^y = \lim\limits_{y \to c} a^{by}$

or $(a^b)^c = a^{bc}$

To show that $(ab)^c = a^c b^c$

$(ab)^y = a^y b^y$

Then $\lim (ab)^y = \lim a^y b^y = \lim a^y . \lim b^y$

That is $(ab)^c = a^c b^c$

* It can be shown that the order in which the variables are allowed to approach their limits does not change the result.

Reference: College Algebra by H. B. Fine

7. The compound interest law and the function $y = e^x$

When $y = e^x$, $\frac{dy}{dx} = e^x$, that is the rate of change of the function is proportional to itself. Money invested at compound interest increases in this way. That property of a function to increase at a rate that is proportional to itself has been called "the compound interest law".

The formula for compound interest when the interest is added annually is $A = P\left(1+\frac{r}{100}\right)^n$ where n is the number of years.

In case the interest is added m times per year the amount at the end of one year is $A = P\left(1+\frac{r}{100m}\right)^m$ and at the end of n years $A = P\left(1+\frac{r}{100m}\right)^{mn}$

To find an expression for the amount when the interest is added continuously we need the limit of $P\left(1+\frac{r}{100m}\right)^{mn}$ when m becomes infinite.

Let $\frac{r}{100m} = \frac{1}{u}$, then $m = \frac{ru}{100}$

At the end of one year $A = P\left[\left(1+\frac{1}{u}\right)^u\right]^{\frac{r}{100}}$

But $\lim_{u\to\infty}\left(1+\frac{1}{u}\right)^u = e$

Let $\frac{r}{100} = a$, then $A = Pe^a$

At the end of n years $A = Pe^{an}$ or $A = Pe^{\frac{rn}{100}}$

Reference: Higher Mathematics for Students of Chemistry and Physics by J. W. Mellor
Differential and Integral Calculus by W. F. Osgood

Thus when the interest is compounded continuously the formula for compound interest is in the form of the function $y = e^x$. When $A = Pe^{an}$, $\frac{dA}{dn} = aPe^{an}$ and the rate of increase of the function is porportional to itself.

If \$500 is invested at 4% for 10 years at compound interest, the interest to be added annually, then

$A = 500\ (1 + .04)^{10}$

$A = \$739.66$

If the interest is added continuously

$A = 500\ e^{.4}$ or $A = \$746.00$

8. The Logarithmic Spiral.

The curve $r = ae^{\lambda\theta}$ where $a > o$ and $\lambda \neq o$, is a spiral which coils around the origin infinitely often.

Since $\frac{dr}{d\theta} = a\lambda e^{\lambda\theta}$ the rate of change of the function is proportional to itself.

One property of the curve is that the tangent always makes the same angle with the radius vector. If ψ is the angle between the tangent and the radius vector, then by a formula from the calculus $\tan \psi = r \frac{d\theta}{dr}$

References: Differential and Integral Calculus by W. F. Osgood

Since $r = ae^{\lambda\theta}$, differentiating

$$dr = a\lambda e^{\lambda\theta} d\theta$$

$$\frac{d\theta}{dr} = \frac{1}{a\lambda e^{\lambda\theta}}$$

Then $\tan \psi = \frac{ae^{\lambda\theta}}{a\lambda e^{\lambda\theta}} = \frac{1}{\lambda}$, a constant.

The equation and curve of the logarithmic spiral appear in various branches of science and in nature. The following are examples.

The law of radioactive decay is expressed by an equation in the form of the logarithmic spiral.

A radioactive substance gradually diminishes, due to disintegration, and the rate of change is proportional to the quantity of changing substance. Half of any given sample passes out of existence in a definite period of time and half of what remains, in the next equal period.

For a single radio-element free from its parent and therefore not being reproduced, the quantity in existence diminishes according to the equation

$$\frac{I_t}{I_o} = e^{-\lambda t}$$

In this equation I_o is the initial quantity, I_t the quantity after any given time t , λ a constant known as the radioactive constant, and e the number 2.71828.

References: General Chemistry by H. G. Deming
The Chemistry of the Radio-Elements by Frederick Soddy.

The radioactive constant differs for different substances. For radium $\lambda = \frac{1}{2440}$

The spiral curve is seen in the orb webs that are built by certain types of spiders. The web consists in part of radii that extend from the center. These are held in place by a spiral line which extends over the entire area that is to be occupied by the web. The turns of the spiral are as far apart as the spider can conveniently reach and the line crosses the radii at right angles. This line is a foundation line and only very small pieces of it appear in the finished web.

The most important part of the web is a spiral that is started on the outer margin of the web and built up toward the center. The spiral consists of a series of short straight lines fastened to the radii. The angles between the lines and the radii are equal. This suggests the logarithmic spiral since in that curve the angles between the radii and the tangents are equal. The turns of the spiral are quite close together. The spiral that served as a foundation for the web is cut away as the second spiral is built up.

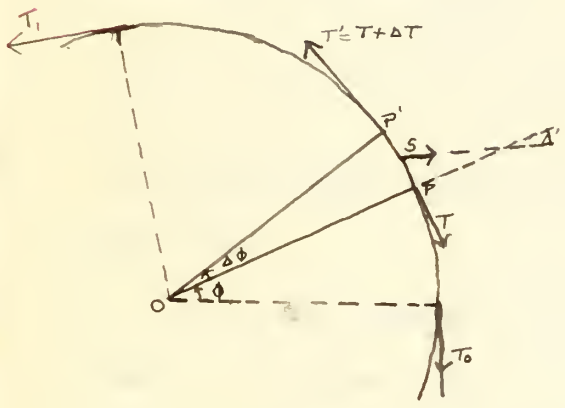
The shell of the chambered nautilus has the form of a logarithmic spiral. The rate of growth of the shell at any time is proportional to its size. Therefore, two parts of the shell have the same shape and differ only in size

References: The Spider Book by J. H. Comstock
A Synoptic Text Book of Zoology by A. W. Weyesse

The shell consists of a series of chambers separated by partitions. The animal occupies one chamber after another. As it grows and requires more room it moves forward into the next chamber, sealing the one it has just occupied. All of the chambers are formed in this way. The sealed chambers are filled with gas which makes the specific gravity of the animal about like that of the surrounding water.

9. The Problem of a Rope round a Post.

Let it be required to find the law of tension in a rope which is wound round a post and is just on the point of slipping.



We are dealing with a flexible, inextensible, weightless string wound on a rough circular drum. Consider the arc PP' and the forces acting on it. They are: the tension T at P, the tension $T' = T+\Delta T$ at P', and the reaction of the drum, which is a force S, inclined toward the direction

References: A Synoptic Text-Book of Zoology by A. W. Weyesse
The Curves of Life by T. A. Cook
Advanced Calculus by W. F. Osgood

opposite to that in which the string tends to slip. Let λ' be the angle between S and the radius OP produced and let λ be the angle between S and the normal to the arc PP'. Then λ is the angle of friction, or $\mu = \tan\lambda$ and $\lim_{P'\to P} \lambda' = \lambda$

Since PP' is in equilibrium, if it becomes rigid it will still be in equilibrium under the action of the forces named above. Then the sum of the components of the forces along any chosen direction must equal zero.

Resolving the forces along the tangent and normal at P, we have

$$T' \cos\triangle\phi = T + S \sin\lambda'$$

$$T' \sin\triangle\phi = S \cos\lambda'$$

Dividing, $\dfrac{T' \cos\triangle\phi - T}{T' \sin\triangle\phi} = \tan\lambda'$

or $T' \cos\triangle\phi - T = T' \tan\lambda' \sin\triangle\phi$

The left-hand side of this equation may be written in the form:

$$(T + \triangle T)\cos\triangle\phi - T = \triangle T \cos\triangle\phi - T(1 - \cos\triangle\phi).$$

Then, $\triangle T \cos\triangle\phi - T(1 - \cos\triangle\phi) = T' \tan\lambda' \sin\triangle\phi$

Divide through by $\triangle\phi$ and take limits:

$$\frac{\triangle T}{\triangle\phi} \cos\triangle\phi - T\,\frac{1 - \cos\triangle\phi}{\triangle\phi} = T' \tan\lambda' \,\frac{\sin\triangle\phi}{\triangle\phi}$$

$$\left(\lim_{\triangle\phi\to 0} \frac{\triangle T}{\triangle\phi}\right)\left(\lim_{\triangle\phi\to 0} \cos\triangle\phi\right) - T \lim_{\triangle\phi\to 0} \frac{1 - \cos\triangle\phi}{\triangle\phi}$$

Reference: Advanced Calculus by W. F. Osgood

$$= \left(\lim_{\Delta\phi\to 0} T'\right)\left(\lim_{\Delta\phi\to 0} \tan\lambda'\right)\left(\lim_{\Delta\phi\to 0} \frac{\sin\Delta\phi}{\Delta\phi}\right)$$

In the calculus it is shown that

$$\lim_{\Delta\phi\to 0} \frac{1-\cos\Delta\phi}{\Delta\phi} = 0 \qquad \text{and} \qquad \lim_{\Delta\phi\to 0} \frac{\sin\Delta\phi}{\Delta\phi} = 1$$

In the first part of the discussion we saw that

$$\lim_{\Delta\phi\to 0} \tan\lambda' = \tan\lambda = \mu$$

Therefore the expression above reduces to $\frac{dT}{d\phi} = \mu T$

Separating the variables

$$\frac{dT}{T} = \mu\, d\phi$$

Then $\int \frac{dT}{T} = \int \mu\, d\phi$

$$\log T = \mu\phi + C$$

When $\phi = 0$, $T = T_0$, therefore $C = \log T_0$

Then $\log T = \mu\phi + \log T_0$

or $T = T_0 e^{\mu\phi}$, which is the law of tension

The result is independent of the radius of the drum.

10. Definition of a complex imaginary exponent.

The complex number $z = x + iy$ can be written in the polar form. Since $x = r\cos\theta$ and $y = r\sin\theta$, $z = r(\cos\theta + i\sin\theta)$ where r is the absolute value of z, that is $|z| = r = \sqrt{x^2+y^2}$

Reference: Advanced Calculus by W. F. Osgood

To find the meaning of e^z consider the following series:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Let $x = \theta i$ in the first series and reduce the result by using the relations $i^2 = -1$, $i^3 = -i$ $i^4 = 1$

Then $e^{\theta i} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots - i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots)$

But these are the series for $\cos \theta$ and $\sin \theta$.

Therefore we define $e^{\theta i}$ as follows: $e^{\theta i} = \cos \theta + i \sin \theta$

Since $e^z = e^{x+yi} = e^x \cdot e^{yi}$

Then $e^z = e^x (\cos y + i \sin y)$

It can be shown that e^z like e^x can be developed in series and that $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$

Another form for z can be given now. Since

$z = r(\cos \theta + i \sin \theta)$, then $z = re^{\theta i}$

Reference: Advanced Calculus by W. F. Osgood

11. The addition theorem for the exponential theorem.

To show that $e^{z_1} e^{z_2} = e^{z_1+z_2}$

$$e^{z_1} e^{z_2} = e^{x_1}(\cos y_1 + i \sin y_1)\, e^{x_2}(\cos y_2 + i \sin y_2)$$

$$= e^{x_1+x_2}\left[(\cos y_1 \cos y_2 - \sin y_1 \sin y_2) + i (\sin y_1 \cos y_2 + \cos y_1 \sin y_2)\right]$$

$$= e^{x_1+x_2}\left[\cos (y_1+y_2) + i \sin (y_1+y_2)\right]$$

But this expression is by definition

$$e^{(x_1+x_2)+i(y_1+y_2)} \quad \text{or} \quad e^{z_1+z_2}$$

Therefore $e^{z_1} e^{z_2} = e^{z_1+z_2}$

12. The period of e^z where z is a complex number.

The exponential function is a periodic function with the imaginary period $2\pi i$

For by definition

$$e^z = e^{x+iy} = e^x \cos y + i\, e^x \sin y$$

Then $e^{z+2k\pi i} = e^{x+i(y+2\pi k)} = e^x \cos(y+2\pi k)\, ie^x \sin(y+2\pi k)$

But $\cos (y+2\pi k) = \cos y$ and $\sin (y+2\pi k) = \sin y$

Therefore $e^z = e^{z+2k\pi i}$ where k is an integer

Reference: Advanced Calculus by W.F. Osgood

Reference: Advanced Calculus by F. S. Woods

13. The function log z

If $z = e^w$ then by definition $w = \log z$.

The logarithm of a complex number is itself a complex number for,

$z = x + iy = r(\cos \theta + i \sin \theta) = re^{i\theta}$

then $\log z = \log r + \log e^{i\theta}$

$= \log r + i\theta$ where θ is any value of the angle which belongs to z.

Since $z = e^w = e^{w+2ki\pi}$

Then $\log z = w+2ki\pi$ where $k = o$ or an integer

Therefore in the domain of the complex numbers the logarithm of a number is not a unique quantity. It has an infinite number of values which differ by multiples of $2\pi i$.

We may now find the logarithm of a real negative number, -a.

For, $-a = a(\cos\pi + i \sin\pi) = ae^{i\pi}$

Therefore $\log(-a) = \log a + i\pi + 2k\pi i$

Then $\log(-1) = i\pi + 2k\pi i$

That is $\log(-1) = i\pi$, or $-i\pi$, or $3i\pi$ etc.

Also, $\log 1 = \log 1 + 2\pi ki$

$= o$, or $\pm 2\pi i$, or $\pm 4\pi i$

The relations $\log z_1 + \log z_2 = \log(z_1 z_2)$ and $\log z^n = n \log z$ hold for proper values of k in the period $2k\pi i$.

References: Advanced Calculus by F. S. Woods and
Advanced Calculus by W. F. Osgood

To find all values of log (2-3i)

$x = 2$, $y = -3$, $r = \sqrt{x^2+y^2} = \sqrt{13}$

$\log z = \log r + i\theta + 2k\pi i$

Then $\log (2-3i) = \log \sqrt{13} + i \tan^{-1}(\frac{-3}{2}) + 2k\pi i$

$= 1.2825 - .9828i + 2k\pi i$

The generalized power A^B where A and B are any two complex numbers , $A \neq 0$.

Since $e^{\log A} = A$, then by definition

$A^B = e^{B \log A}$, where log A has any one of its possible values.

For example $i^i = e^{i \log i}$

$\log i = \log \sqrt{-1} = \frac{1}{2} \log (-1)$

But $\log (-1) = i\pi + 2k\pi i = \pi i$, $-\pi i$, $3\pi i$, $5\pi i$, $-3\pi i$

Then $i^i = e^{\frac{-\pi}{2}}$, or $e^{\frac{\pi}{2}}$, or $e^{\frac{3\pi}{2}}$ - - - - - or $e^{\frac{-\pi}{2}+2k\pi}$

References: Advanced Calculus by F. S. Woods and

Advanced Calculus by W. F. Osgood

14. The functions sin z , cos z , tan z

By definition $e^{\theta i} = \cos\theta + i \sin\theta$ where θ is real

then $e^{\theta i} = \cos\theta - i \sin\theta$

Subtracting $\sin\theta = \dfrac{e^{\theta i} - e^{-\theta i}}{2i}$

Adding $\cos\theta = \dfrac{e^{\theta i} + e^{-\theta i}}{2}$

If the definition is extended to include complex values of θ the functions become

$$\sin z = \frac{e^{iz} - e^{iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

We can show that the important properties of the trigonometric functions hold for these extended definitions. One of the most important properties is the addition theorem. If that theorem still holds then,

$$\sin (z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$

$$\cos (z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

Computing the right hand side of the first equation from the definitions and using the fact that $e^{z_1} e^{z_2} = e^{z_1 + z_2}$

We get,

Reference: Advanced Calculus by W. F. Osgood

$$\frac{e^{iz_1}-e^{-iz_1}}{2i}\cdot\frac{e^{iz_2}+e^{-iz_2}}{2}+\frac{e^{iz_1}+e^{-iz_1}}{2}\cdot\frac{e^{iz_2}-e^{-iz_2}}{2i}=$$

$$\frac{1}{4i}\Big(e^{iz_1+iz_2}-e^{-iz_1+iz_2}+e^{iz_1-iz_2}-e^{-iz_1-iz_2}+e^{iz_1+iz_2}+e^{-iz_1+iz_2}$$

$$-e^{iz_1-iz_2}-e^{-iz_1-iz_2}\Big)=$$

$$\frac{1}{4i}\left[2\,e^{i(z_1+z_2)}-2e^{-i(z_1+z_2)}\right]=\frac{e^{i(z_1+z_2)}-e^{-i(z_1+z_2)}}{2i}$$

But that expression is exactly the value of $\sin(z_1+z_2)$ by definition. Therefore the addition theorem for the trigonometric functions holds when the angle , θ , is a complex quantity.

The real functions sin x , cos x, etc. have derivatives. It will be shown later that the functions sin z, cos z, etc. also have derivatives and that

$$\frac{d\sin z}{dz}=\cos z\ ,\ \text{and}\ \frac{d\cos z}{dz}=-\sin z$$

and therefore the functions sin z and cos z are solutions of the differential equation $\frac{d^2w}{dz^2}+w=0$

It can be shown also that

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + - - - -$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + - - - -$$

The other trigonometric functions are defined in terms of sin z , cos z

$$\tan z = \frac{\sin z}{\cos z} = \frac{1}{i} \cdot \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}$$

To compute sin (1 + i)

$$\sin (1+i) = \frac{e^{i(1+i)} - e^{-i(1+i)}}{2i} = \frac{e^{i-1} - e^{-i+1}}{2i}$$

$e^{i-1} = e^{-1}(\cos 1 + i \sin 1)$ and $e^{-i+1} = e\big(\cos(-1) + \sin(-1)\big)$

$$\sin (1+i) = \frac{e^{-1}(\cos 1 + i \sin 1) - e(\cos 1 - i \sin 1)}{2i}$$

$$\sin (1+i) = \frac{.19875 + .3095i - 1.469 + 2.2873}{2i}$$

$$\sin (1+i) = \frac{-1.27025 + 2.5968i}{2i}$$

$$\sin (1+i) = .63512i + 1.2984$$

Reference: Advanced Calculus by W. F. Osgood

15. The inverse trigonometric functions $\sin^{-1} z$, $\cos^{-1} z$, $\tan^{-1} z$.

The anti - sine is defined by the equations

$$w = \sin^{-1} z \quad \text{if } \sin w = z.$$

To find w the second equation may be written as

$$\frac{e^{iw} - e^{-iw}}{2i} = z$$

Then $e^{iw} - e^{-iw} - 2 i z = 0$

Multiplying by e^{iw}

$$e^{2iw} - 2 i z e^{iw} - 1 = 0$$

Solving $e^{iw} = \frac{2 i z \pm \sqrt{-4 z^2+4}}{2}$

or $e^{iw} = iz \pm \sqrt{1-z^2}$

Then $iw = \log (iz \pm \sqrt{1-z^2})$

or $w = \frac{1}{i} \log (iz \pm \sqrt{1-z^2})$ or $-i \log (iz \pm \sqrt{1-z^2})$

Since $- \log (iz \pm \sqrt{1-z^2}) = \log \frac{1}{(iz \pm \sqrt{1-z^2})}$

and $\frac{1}{iz \pm \sqrt{1-z^2}} = \pm \sqrt{1-z^2} - iz$

We may write $\sin^{-1} z = i \log (\pm \sqrt{1-z^2} - iz)$

Reference : Advanced Calculus by W. F. Osgood

The anti - cosine is defined by the equations

$$w = \cos^{-1} z \quad \text{if} \quad \cos w = z$$

Then $\frac{e^{iw} + e^{-iw}}{2} = z$

$$e^{iw} + e^{-iw} - 2z = 0$$

Multiplying by e^{iw}

$$e^{2iw} - 2ze^{iw} + 1 = 0$$

$$e^{iw} = \frac{2z \pm \sqrt{4z^2 - 4}}{2}$$

$$e^{iw} = z \pm \sqrt{z^2 - 1}$$

Then $w = \frac{1}{i} \log (z \pm \sqrt{z^2 - 1}) = -i \log (z \pm \sqrt{z^2 - 1}$

$$-\log (z \pm \sqrt{z^2 - 1}) = \log \frac{1}{z \pm \sqrt{z^2 - 1}}$$

$$\frac{1}{z \pm \sqrt{z^2 - 1}} = \frac{i}{\pm\sqrt{1 - z^2} + iz} \cdot \frac{\pm\sqrt{1 - z^2} - iz}{\pm\sqrt{1 - z^2} - iz} = \pm i\sqrt{1 - z^2} + z$$

Then $\cos^{-1} z = i \log (z \pm i\sqrt{1 - z^2})$.

The anti - tangent is defined by the equations

$w = \tan^{-1} z$ is $\tan w = z$

Reference: Advanced Calculus by W. F. Osgood

Then $\frac{1}{i} \cdot \frac{e^{iw}-e^{-iw}}{e^{iw}+e^{-iw}} = z$

or $z = -i \cdot \frac{e^{2iw}-1}{e^{2iw}+1}$

$e^{2iw}z + z = -ie^{2iw} + i$

$e^{2iw}(i+z) = i - z$

$e^{2iw} = \frac{1-z}{1+z}$

$w = \frac{1}{2i} \log \frac{1-z}{1+z} = \frac{-i}{2} \log \frac{1-z}{1+z}$

But $-\log \frac{1-z}{1+z} = \log \frac{1}{\frac{1-z}{1+z}} = \log \frac{1+z}{1-z}$

Then $\tan^{-1} z = \frac{i}{2} \log \frac{1+z}{1-z}$

To compute $\sin^{-1} 2$.

$\sin^{-1} 2 = i \log (\pm \sqrt{-3} - 2i)$

$= i \log \left[i(\pm \sqrt{3} - 2) \right]$

$= i \left[\log i + \log (\pm\sqrt{3} - 2) \right]$

$= i \left[\log i + \log (-.268) \right]$ or $i \left[\log i + \log (-3.732) \right]$

$= i \left[\frac{(2k+1)\pi i}{2} - 1.317 + (2k+1)\pi i \right]$

Reference: Advanced Calculus by W. F. Osgood

$$\text{or} \quad i\left[\frac{(2k+1)\pi i}{2} + 1.31694 + (2k+1)\pi i\right]$$

$$\sin^{-1} 2 = -\,{}^{3}/_{2}\,(2k+1)\pi - 1.317i \quad \text{or}$$

$$\sin^{-1} 2 = -\,{}^{3}/_{2}\,(2k+1)\pi + 1.31694\,i$$

To compute $\tan^{-1}(1+i)$

$$\tan^{-1}(1+i) = \frac{i}{2}\log\frac{1+2i}{-1}$$

$$= \frac{i}{2}\log(-1-2i)$$

$$\log(-1-2i) = {}^{1}/_{2}\log 5 + i\tan^{-1}\frac{2}{1}$$

$$= .80472 + 4.2488i + 2k\pi i$$

$$\tan^{-1}(1+i) = \frac{i}{2}(.80472 + 4.2488i + 2k\pi i)$$

$$\tan^{-1}(1+i) = .40236i - 2.1244 - k\pi$$

Reference: Advanced Calculus by W. F. Osgood

16. The Hyperbolic Functions

The hyperbolic functions are analogous to the trigonometric functions and are defined as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

The functions satisfy the following relations:

$\sinh(-x) = -\sinh x$, $\cosh(-x) = \cosh x$, $\tanh(-x) = -\tanh x$

Also $\sinh(o) = o$, $\cosh(o) = 1$, $\tanh(o) = o$,

$\cosh^2 x - \sinh^2 x = 1$ and $1 - \tanh^2 x = \operatorname{sech}^2 x$

The hyperbolic sine and cosine are connected with the trigonometric sine and cosine by the relations

$\sin i x = i \sinh x$ and $\cos i x = \cosh x$ for,

$$\sin x = \frac{e^{xi} - e^{-xi}}{2i}$$

$$\sin i x = \frac{e^{-x} - e^{x}}{2i}$$

Multiplying by $\frac{i}{i}$, $\sin i x = i \cdot \frac{e^x - e^{-x}}{2} = i \sinh x$

In the same way it can be shown that $\cos i x = \cosh x$

The addition theorems for the hyperbolic functions are as follows:

References: Advanced Calculus by F. S.Woods
Advanced Calculus by W. F. Osgood

$$\sinh (x+y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh (x+y) = \cosh x \cosh y + \sinh x \sinh y$$

$$\tanh (x+y) = \frac{\tanh x + \tanh y}{1+\tanh x \tanh y}$$

These formulas may be obtained by substituting $\sin i x = i \sinh x$ and $\cos i x = \cosh x$ in the addition formulas of trigonometry.

The derivatives of the hyperbolic functions have the values:

$$\frac{d \sinh x}{dx} = \frac{e^{x}+e^{-x}}{2} = \cosh x$$

$$\frac{d \cosh x}{dx} = \sinh x$$

$$\frac{d \tanh x}{dx} = \operatorname{sech}^2 x$$

The inverse of the hyperbolic sine is defined by the equation $y = \sinh^{-1} x$ if $x = \sinh y$.

Then $x = \frac{e^{y}-e^{-y}}{2}$

Solving $e^{y} = x \pm \sqrt{1+x^2}$

Since $e^{y} > o$ for all values of y , only the positive sign of the radical is possible.

Then $y = \log (x+\sqrt{1+x^2})$

That is, $\sinh^{-1} x = \log (x+\sqrt{1+x^2})$

References: Advanced Calculus by F. S. Woods
Advanced Calculus by W. F. Osgood

In the same way it can be shown that

$$\cosh^{-1} x = \log (x \pm \sqrt{x^2 - 1}) \qquad x \geq 1$$

The upper sign corresponds to positive values of $\cosh^{-1} x$.

Also $\tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x} \qquad -1 < x < 1$

From the equation $x = \sinh y$ we get,

$dx = \cosh y \, dy$

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}}$$

That is $\frac{d \sinh^{-1} x}{dx} = \frac{1}{\sqrt{1+x^2}}$ where the plus sign is taken for the radical when x is real because by definition $\cosh y$ is always positive.

Similarly, $\frac{d \cosh^{-1} x}{dx} = \pm \frac{1}{\sqrt{x^2 - 1}}$

$$\frac{d \tanh^{-1} x}{dx} = \frac{1}{1 - x^2}$$

The Hyperbolic Functions of a Complex Variable.

The definitions of the hyperbolic sine and cosine for a complex variable are the same as those for a real variable.

That is, $\sinh z = \frac{e^z - e^{-z}}{2}$ and

$$\cosh z = \frac{e^z + e^{-z}}{2}$$

References: Advanced Calculus by F. S. Woods

Advanced Calculus by W. F. Osgood

Then $\cosh^2 z - \sinh^2 z = 1$

From the definitions we get

$$\sinh i z = \frac{e^{iz} - e^{-iz}}{2} = i \sin z$$

$$\cosh i z = \frac{e^{iz} + e^{-iz}}{2} = \cos z$$

Also $$\sinh z = \frac{e^{-i(iz)} - e^{i(iz)}}{2} = -i \sin i z$$

$$\cosh z = \frac{e^{-i(iz)} + e^{i(iz)}}{2} = \cos i z$$

Then $\sinh (z_1 + z_2) = -i \sin (iz_1 + iz_2)$

$= -i \sin i z_1 \cos i z_2 - i \cos i z_1 \sin i z_2$

$= \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$

$\cosh (z_1 + z_2) = \cos (iz_1 + iz_2)$

$= \cos i z_1 \cos i z_2 - \sin i z_1 \sin i z_2$

$= \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$

Applying the addition theorem

$$\sinh (z + 2k\pi i) = \sinh z \cosh 2k\pi i + \cosh z \sinh 2k\pi i$$

But $\cosh 2k\pi i = \cos 2k\pi = 1$

And $\sinh 2k\pi i = i \sin 2k\pi = 0$

Therefore $\sinh (z + 2k\pi i) = \sinh z$

Similarly $\cosh (z + 2k\pi i) = \cosh z$

Hence in the case of complex variables the hyperbolic sine and the hyperbolic cosine are periodic functions with the imaginary period $2\pi i$.

References: Advanced Calculus by F. S. Woods
Advanced Calculus by W. F. Osgood

The Inverse Hyperbolic Functions.

If $z = \sinh w$ then, $w = \sinh^{-1} z$

$$z = \frac{e^w - e^{-w}}{2}$$

Solving, $e^w = z \pm \sqrt{z^2+1}$

Then $w = \sinh^{-1} z = \log (z \pm \sqrt{z^2+1})$

If $z = x$, a real number, then this formula gives only one real value for w obtained by using the positive value of the radical.

It can be shown that $\cosh^{-1} z = \log (z \pm \sqrt{z^2 - 1})$ If $z = x$, a real number, we have two real values of w provided $x > 1$.

Also $\tanh^{-1} z = \frac{1}{2} \log \frac{1-z}{1+z}$

If $z = x$ we have one real value of w when $-1 < x < 1$

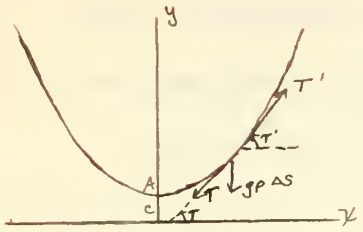
These formulas are true for any complex quantity z.

17. The Catenary.

The catenary is the curve in which a chain hangs. We wish to find the equation of the curve.

It is assumed that the chain is flexible, inextensible, and of constant density, ρ. Its ends are fastened at two fixed points and the chain hangs at rest under the force of gravity.

References: Advanced Calculus by F. S. Woods
Advanced Calculus by W. F. Osgood



Let the axis of x be horizontal, and at a distance C below the lowest point , A, of the curve. Let the axis of y pass through A. Let P (x,y) be any point of the curve, and P'(x+Δx, y+Δy) be a second point. Consider the arc PP' and the forces which act on it. They are (1) the force of gravity gρΔs where ρ is the mass of one unit of length of the chain; (2) the tension T at P, directed down the tangent; (3) the tension T' = T+ΔT at P' , directed up the tangent.

Since PP' is in equilibrium, if it becomes rigid it will still be in equilibrium under the action of the forces named above. Then the sum of the components of these forces along any direction must equal zero.

Resolving horizontally and vertically, we have:

$$T' \cos\tau' = T \cos\tau$$

$$T' \sin\tau' = T \sin\tau + g\rho\Delta s$$

If we take P' at A, then $T' = T_o$, where T_o is the tension at the lowest point , and $\tau' = o$

Then the first equation becomes

$$T \cos\tau = T_o \qquad \text{or} \qquad T = T_o \sec\tau$$

The second equation can be written in the form:

$$T' \sin\tau' - T \sin\tau = g\rho\Delta s$$

Reference: Advanced Calculus by W. F. Osgood

The left-hand side of this equation is the increment of the function T sin τ . Therefore the equation becomes

$$\Delta(T \sin\tau) = g\rho\,\Delta s$$

Divide through by Δx and let Δx approach the limit zero.

$$\lim_{\Delta x \to 0} \frac{\Delta(T \sin\tau)}{\Delta x} = g\rho \lim_{\Delta x \to 0} \frac{\Delta s}{\Delta x}$$

or $$D_x (T \sin\tau) = g\rho D_x s$$

Replacing T by its value $T_o \sec\tau$, we have:

$$T_o D_x (\tan\tau) = g\rho D_x s$$

or, since $\tan\tau = D_x y$

$$D_x^2 y = \frac{1}{C} D_x s \quad , \quad \text{where } C = \frac{T_o}{g\rho}$$

But $$D_x s = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Therefore the equation may be written in the form:

$$\frac{d^2y}{dx^2} = \frac{1}{C}\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Let $p = \frac{dy}{dx}$ and substitute

Then, $$\frac{dp}{dx} = \frac{\sqrt{1+p^2}}{c}$$

Separating the variables,

$$\frac{dx}{c} = \frac{dp}{\sqrt{1+p^2}} \quad \text{or} \quad \frac{x}{c} = \int \frac{dp}{\sqrt{1+p^2}} = \log\left(p+\sqrt{1+p^2}\right)+C$$

Reference : Advanced Calculus by W. F. Osgood

At A , $x = o$ and $p = o$. Therefore $C = o$

Then $\frac{x}{c} = \log (p + \sqrt{1 + p^2})$

or $\sqrt{1 + p^2} + p = e^{\frac{x}{c}}$ (1)

To solve this equation for P take the reciprocal of each side. Then $\frac{1}{\sqrt{1 + p^2} + p} = e^{\frac{-x}{c}}$

Rationalize the denominator and we get

$\sqrt{1 + p^2} - p = e^{\frac{-x}{c}}$

Subtract this equation from (1)

$2p = e^{\frac{x}{c}} - e^{\frac{-x}{c}}$ or $\frac{dy}{dx} = \frac{1}{2} (e^{\frac{x}{c}} - e^{\frac{-x}{c}})$

Then $y = \int \frac{1}{2} (e^{\frac{x}{c}} - e^{\frac{-x}{c}}) \, dx = \frac{c}{2} (e^{\frac{x}{c}} + e^{\frac{-x}{c}}) + K$

When $x = o$, $y = c$ and therefore $K = 0$

Then the equation of the catenary is

$$Y = \frac{c}{2} (e^{\frac{x}{c}} + e^{\frac{-x}{c}})$$

When $C = 1$, we have the hyperbolic cosine,

$y = \cosh x$ and in general

$y = c \cosh \frac{x}{c}$

Reference: Advanced Calculus by W. F. Osgood

18. The Cauchy - Riemann differential equations.

To obtain the necessary conditions that the function $w = f(z)$ have a derivative, let

$w = u + vi$ and $z = x + yi$

Let Δz approach zero passing through real values, then $\Delta y = 0$ and $\Delta z = \Delta x$

$$\text{Then } \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta x \to 0} \frac{\Delta_x u}{\Delta x} + i \frac{\Delta_x v}{\Delta x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

$$\text{therefore } D_z w = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

Now let Δz approach zero passing through pure imaginary values, then $\Delta x = 0$ and $\Delta z = i \Delta y$

$$\text{Then } \lim_{\Delta z \to 0} \frac{\Delta u}{\Delta z} = \lim_{\Delta y \to 0} \left(\frac{1}{i} \frac{\Delta_y u}{\Delta y} + \frac{\Delta_y v}{\Delta y} \right) = \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} =$$

$$\frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

$$\text{Therefore } D_z w = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

Equating these two values of $D_z w$

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

Since these two expressions are equal, the real parts must be equal.

$$\text{Therefore } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Reference: Advanced Calculus by W. F. Osgood

The result may also be written in the form of one equation in complex quantities $\frac{\partial w}{\partial x} = \frac{1}{i} \frac{\partial w}{\partial y}$

These are the necessary conditions that the function $w = f(z)$ have a derivative.

If we assume that the real functions $u = u(x, y)$ and $v = v(x, y)$ are continuous together with their first partial derivatives then the equations above form a sufficient condition that $w = u + iv$ have a derivative.

For, $$\Delta u = \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

$$\Delta v = \frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y + \epsilon_1' \Delta x + \epsilon_2' \Delta y$$

where the ϵ's are infinitesimals, Δx and Δy being the principal infinitesimals. Form the quotient

$$\frac{\Delta w}{\Delta z} = \frac{\Delta u + i \Delta v}{\Delta x + i \Delta y}$$

Now, $$\left(\frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y\right) + i \left(\frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y\right) = \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}\right)(\Delta x + i \Delta y)$$

For, $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Substituting in the expression above and factoring out Δx and $i \Delta y$

Reference: Advanced Calculus by W.F. Osgood

$$\Delta x \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}\right) + i \Delta y \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}\right)$$

$$= \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}\right)(\Delta x + i \Delta y)$$

Therefore $\frac{\Delta w}{\Delta z} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} + (\epsilon_1 + \epsilon_1') \frac{\Delta x}{\Delta z} + (\epsilon_2 + \epsilon_2') \frac{\Delta y}{\Delta z}$

$|\Delta x| \leq |\Delta z|$ and $|\Delta y| \leq |\Delta z|$ therefore $\left|\frac{\Delta x}{\Delta z}\right| \leq 1$ and $\left|\frac{\Delta y}{\Delta z}\right| \leq 1$. The expressions $\epsilon_1 + \epsilon_1'$ and $\epsilon_2 + \epsilon_2'$ are infinitesimals and they approach zero as Δz approaches zero. Since one quantity approaches zero and the other remains finite the product approaches zero, that is, $\lim_{\Delta z \to 0} (\epsilon_1 + \epsilon_1') \frac{\Delta x}{\Delta z} = 0$ and $\lim_{\Delta z \to 0} (\epsilon_2 + \epsilon_2') \frac{\Delta y}{\Delta z} = 0$. Therefore $\frac{\Delta w}{\Delta z}$ approaches a limit when Δz approaches zero and the function w has a derivative. The value of the derivative is given by the formula

$$D_z w = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

To show that sin z has a derivative given by the formula $\frac{d \sin z}{dy} = \cos z$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{ix-y} - e^{-ix+y}}{2i}$$

But $e^{ix} = \cos x + i \sin x$ and

$e^{-ix} = \cos x - i \sin x$

Therefore $\sin z = \frac{e^{-y}\cos x + ie^{-y}\sin x - e^{y}\cos x + ie^{y}\sin x}{2i}$

Reference: Advanced Calculus by W. F. Osgood

$$\sin z = \frac{1}{2}(-ie^{-y}\cos x + e^{-y}\sin x + ie^{y}\cos x + e^{y}\sin x)$$

$$u = \frac{e^{-y}\sin x + e^{y}\sin x}{2}$$

$$v = \frac{-e^{-y}\cos x + e^{y}\cos x}{2}$$

$$\frac{\partial u}{\partial x} = \frac{e^{-y}\cos x + e^{y}\cos x}{2}$$

$$\frac{\partial v}{\partial y} = \frac{e^{-y}\cos x + e^{y}\cos x}{2}$$

Therefore $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$$\frac{\partial u}{\partial y} = \frac{-e^{-y}\sin x + e^{y}\sin x}{2}$$

$$\frac{\partial v}{\partial x} = \frac{e^{-y}\sin x - e^{y}\sin x}{2}$$

Therefore $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ the function sin z has a derivative. The derivative is given by the formula $D_z W = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

Reference: Advanced Calculus by W. F. Osgood

$$\frac{d \sin z}{dz} = \frac{e^{-y}\cos x + e^{y}\cos x + ie^{-y}\sin x - ie^{y}\sin x}{2}$$

$$= \frac{e^{-y}(\cos x + i \sin x) + e^{y}(\cos x - i \sin x)}{2}$$

$$= \frac{e^{-y+ix} + e^{y-ix}}{2}$$

$$= \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2}$$

$$= \frac{e^{iz} + e^{-iz}}{2} = \cos z$$

Therefore $\frac{d \sin z}{dz} = \cos z$

In the same way it can be shown that

$$\frac{d \cos z}{dz} = -\sin z \quad \text{and}$$

$$\frac{d \tan z}{dz} = \sec z$$

Reference: Advanced Calculus by W. F. Osgood

15. Mapping the function $w = e^z$ where $w = u + iv$ and

$z = x + iy$

By definition $e^{x+iy} = e^x(\cos y + i \sin y)$

Then $u + iv = e^x \cos y + ie^x \sin y$

$u = e^x \cos y$

$v = e^x \sin y$

If $x = c_1$, then $u = e^{c_1} \cos y$

$v = e^{c_1} \sin y$

Squaring and adding $u^2 + v^2 = e^{2c_1}$

Therefore the straight lines $x = c_1$, parallel to the axis of imaginaries on the z- plane correspond to the circles $u^2 + v^2 = e^{2c_1}$ with center at the origin. If $c_1 = o$ the radius of the circle is 1. If c_1 is greater than zero the radius is greater than one and if c_1 is less then zero the radius is less than one. Hence the right hand half of the z - plane corresponds to the portion of the w - plane outside the circle of radius one and the left hand side of the z - plane corresponds to the portion of the w - plane inside the same circle.

If $y = c_2$, $u = e^x \cos c_2$ (1)

$v = e^x \sin c_2$ (2)

Divide equation (2) by equation (1)

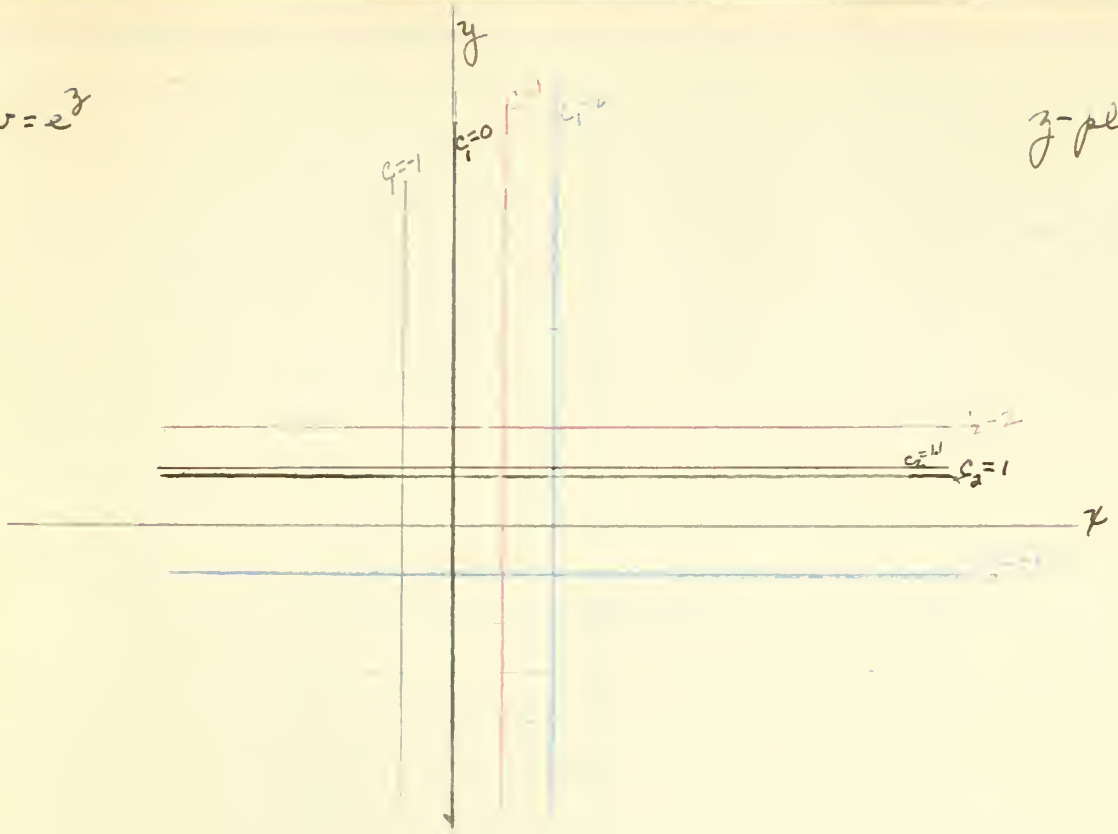
$\frac{v}{u} = \tan c_2$ or $v = u \tan c_2$

Reference:
Advanced Calculus by F. S. Woods

Therefore the straight lines $y = c$, parallel to the axis of reals on the y - plane correspond to the straight lines $v = u \tan c$, on the w - plane which are orthogonal to the circles $u^2 + v^2 = e^{2c}$.

Reference: Advanced Calculus by F. S. Woods

$w = e^z$



z-plane

$x = c_1$

$c_1 = 0$

$c_1 = 1$

$c_1 = 2$

$c_1 = -1$

$y = c_2$

$c_2 = 1$

$c_2 = 2$

$c_2 = -1$

w-plane

$u^2 + v^2 = e^{2c_1}$

$v = u \tan c_2$

SUMMARY

The laws of exponents are valid when the exponents are integral, fractional, positive, negative, rational or irrational numbers.

The compound interest law can be written in the form of the function $y = e^x$ if the interest is added continuously. In that case the amount of P dollars at r per cent interest at the end of n years is given by the formula $A = Pe^{\frac{rn}{100}}$

The equation of the logarithmic spiral is $r = ae^{\lambda\theta}$. The law of radioactive decay is expressed by an equation in this form. The spiral curve is common in nature. Certain types of spiders build webs that have the spiral form. The shell of the chambered nautilus is a logarithmic spiral.

If a rope is wound round a rough circular drum and is just on the point of slipping, the tension in the rope is independent of the radius of the drum and is given by the formula $T = T_0 e^{\mu\theta}$.

In elementary algebra $e^x \cdot e^y = e^{x+y}$ and this law is valid when the exponents are complex imaginary numbers, that is, $e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2}$. This is the addition theorem for the exponential function and is an essential property of that function.

Two other properties of the real function e^x are

$$\frac{de^x}{dx} = e^x \quad \text{and} \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \text{-----}$$

The function e^z also has these properties

$$\frac{de^z}{dz} = e^z \quad \text{and} \quad e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \text{-----}$$

In addition the function e^z has a new property. It is periodic in the imaginary domain and has the period $2\pi i$ for $e^z = e^{z+2\pi i}$.

If $e^w = z$ then by definition $w = \log z$. The real function $\log x$ is single-valued and defined for positive values of x only but the function $\log z$ is defined for all complex values of z, except zero, and is infinitely multiple-valued.

The generalized power A^B where A and B are any two complex numbers provided A is not zero, is defined as

$$A^B = e^{B \log A}.$$

The functions sin z and cos z are exponential functions since by definition $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$ and $\cos z = \frac{e^{iz} + e^{-iz}}{2}$.

An important property of the functions sin x and cos x is given by the addition theorem for these functions. The same theorem applies to the functions sin z and cos z giving the equations

$$\sin (z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$

$$\cos (z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

The functions sin x and cos x have derivatives and power series expansions. The functions sin z and cos z have the same properties and

$$\frac{d \sin z}{dz} = \cos z \quad , \quad \frac{d \cos z}{dz} = - \sin z$$

$$\text{Also, } \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\cdots\cdots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\cdots\cdots$$

The inverse trigonometric functions in the imaginary domain are logarithmic functions.

The hyperbolic functions are exponential functions. The hyperbolic sine and cosine are connected with the trigonometric sine and cosine by the relations $\sin i x = i \sinh x$ and $\cos i x = \cosh x$. The properties of the hyperbolic functions and the relations between them are similar to those of the trigonometric functions.

The catenary is the curve in which a chain hangs when its ends are fastened at two fixed points. The curve is the same as the graph of the function cosh x. The equation of the catenary is

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{\frac{-x}{c}}\right) \quad \text{or} \quad y = c \cosh \frac{x}{c} \quad .$$

The necessary and sifficient conditions that the function $w = f(z)$ have a derivative are that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

The value of the derivative is given by the formula

$$D_z w = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

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